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Renormalisation group studies of the Blume–Emery–Griffiths model in two dimensions

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Abstract. We study a spin-1 Blume–Emery–Griffiths model on the triangular lattice using a generalisation of Niemeijer and van Leeuwen's discrete spin renormalisation group transformation which preserves the Potts symmetry of this model.

We obtain an overall phase diagram that is basically in agreement with other treatments of this model. Our exponents are compared with other renormalisation group studies and some series results.

1. Introduction

Crossover phenomena near tricritical points have been studied extensively from several points of view in the last few years. These include general scaling arguments (Griffiths 1970, 1973, Riedel 1972, Hankey *et al* 1972), series extrapolation techniques (Harbus and Stanley 1973, Saul *et al* 1974), Monte Carlo calculations (Arora and Landau 1973), mean field theories (Blume *et al* 1971) and, more recently, renormalisation group studies at three dimensions (Riedel and Wegner 1972, 1973), at $d = 3 - \epsilon$ dimensions (Stephen and McCauley 1973, Chang *et al* 1974), at two dimensions (Nienhuis and Nauenberg 1976) and at one dimension (Krinsky and Furman 1975).

There have been two distinct discrete spin models which yield tricritical points. One of these is the spin-$\frac{1}{2}$ metamagnet, which is an antiferromagnet which becomes tricritical in a sufficiently large magnetic field, and the second is the spin-1 Blume–Emery–Griffiths (BEG) model, constructed mainly to study ^{3}He–^{4}He mixtures (Blume *et al* 1971). In the continuous spin limit, both models reduce to the standard Ginzburg–Landau–Wilson Hamiltonian (Riedel and Wegner 1972, Nelson and Fisher 1975, Fisher and Nelson 1975). This Hamiltonian can be treated exactly at $d = 3$, to yield a Gaussian tricritical point exhibiting mean-field-like exponents modified by logarithmic corrections (Riedel and Wegner 1972, 1973). Studies of the same model at $d = 3 - \epsilon$ yield ϵ expansions of the critical exponents, which make it possible to try and continue the results to $d < 3$ (Stephen and McCauley 1973, Chang *et al* 1974).

The metamagnet was recently studied at $d = 2$ by Nienhuis and Nauenberg (1976), using discrete spin renormalisation group transformation (Niemeijer and van Leeuwen 1974). They studied the competition between tricritical and critical ordering arising from the crossover between the appropriate two fixed points. In addition they

emphasised the role of an additional fixed point, describing the first-order transition beyond the tricritical point. Among other things, they obtained estimates for the tricritical exponents, which are rather distinct from those of mean field theory.

A priori, there seems to be little relation between the metamagnet and the BEG models at $d = 2$. In this paper, we intend to study the BEG model at $d = 2$, using a discrete spin renormalisation group (Niemeijer and van Leeuwen 1974), and to compare the results with those of the metamagnet. An interesting question we hope to be able to study involves the universality of tricritical points at $d = 2$.

The BEG model (Blume *et al* 1971) is a spin-1 lattice model, defined by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{\langle ij \rangle} [KS_iS_j + LS_i^2S_j^2 + M(S_i^2 + S_j^2)]. \tag{1}$$

The sum is over nearest neighbour pairs, and $S_i = 1, 0$ or -1. The relation of the coefficients K, L and M to the physical parameters of ^{3}He–^{4}He mixtures is discussed at length by Blume *et al* (1971). The same Hamiltonian may describe the behaviour of competing dipolar and quadrupolar orderings in magnets (Chen and Levy 1973), or that of a spin-$\frac{1}{2}$ Ising model with annealed non-magnetic impurities (Wortis 1974).

Several special limits should be noted. For $M \to \infty$ (or for $L \to \infty$) the $S = 0$ state is suppressed, and the model reduces to the spin-$\frac{1}{2}$ Ising model. For $K = 0$, one can define a variable $\sigma_i = 2S_i^2 - 1 = \pm 1$, in terms of which the Hamiltonian again reduces to a spin-$\frac{1}{2}$ Ising model, with an exchange $L/4$ and a temperature dependent magnetic field $[z(M + L/2) + \ln 2]/2$, where z is the coordination number (Griffiths 1967). This limit probably corresponds to quadrupolar ordering (Chen and Levy 1973). For $K = \frac{1}{3}L = -\frac{1}{2}M$, the model reduces to the Potts (1952) model, $-\beta H = 2K \sum_{\langle ij \rangle} (\delta_{s_i s_j} - 1)$ (Joseph and Kim 1974). Any renormalisation group study should lead to fixed points which describe these limiting cases. In addition, we expect of course to find a tricritical fixed point and fixed points at $T = 0$ and $T = \infty$.

An interesting question involves the universality of Ising critical behaviour, i.e. do the spin-1 and the spin-$\frac{1}{2}$ models lead to the same exponents (e.g. Van Dyke and Camp 1975)? In recent work, Berker (1975) studied the spin-S model at $d = 2$ by first projecting it on the spin-$\frac{1}{2}$ one and then following Niemeijer and van Leeuwen (1974). This, of course, does not answer the question of universality. As we shall see, we find that the renormalisation group flow of the Hamiltonian (1), in the critical regime, indeed goes to the Ising $S = \frac{1}{2}$ fixed point, thus proving universality within the present calculation.

There are several possible generalisations of the Niemeijer–van Leeuwen (1974) approach to spin-1 Hamiltonians. We shall discuss some of these, within the two-cell cluster approximation, in the next section. The resulting fixed points and Hamiltonian flows will be described in § 3 and the results will be summarised and discussed in § 4.

After we finished the calculations described in this paper, we received a paper by Berker and Wortis (1976), in which they discuss the BEG model on a square lattice, using a Niemeijer–van Leeuwen (1974) technique. The global phase diagrams, and the Hamiltonian flows which they find are very similar to the ones we find. However, there are several differences in the renormalisation group transformation and in the numerical results (not to mention the different lattice structure; we work on a triangular lattice). We thus decided to emphasise here the differences with the independent Berker–Wortis work, and to refer to their work for the more general picture, which is similar. We will also compare our work with several calculations

which provide partial results for the BEG model. These include the work of Burkhardt (1976) who has applied Kadanoff's variational method (Kadanoff *et al* 1976) to the Blume–Capel model and Burkhardt *et al* (1976) and Dasgupta (1976) who have studied the 3-state Potts model.

2. Renormalisation group transformation

Following Niemeijer and van Leeuwen (1974) we consider a triangular lattice (figure 1), with a spin S_i^α, equal to ± 1 or 0 at each site. We now group the spins into triangular cells, so that the three spins S_i^1, S_i^2, S_i^3 are grouped into the cell i, with cell spin

$$\mu_i = \mu_i(S_i^1, S_i^2, S_i^3). \tag{2}$$

The cell spins μ_i are chosen so as to have again the values ± 1 or 0.

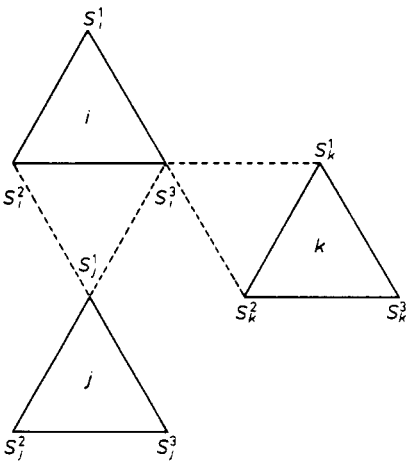

Figure 1. Three cells on the triangular lattice.

The cell spin Hamiltonian $H'\{\mu_i\}$ is found through (Niemeijer and van Leeuwen 1974)

$$\exp(H'\{\mu_i\}) = \sum_{\{S_i^\alpha\}}{}' \exp(H\{S_i^\alpha\}), \tag{3}$$

where the sum Σ' is over all the spin configurations which are consistent with a given cell spin configuration $\{\mu_i\}$.

The transformation (3) cannot be calculated exactly, and there are various possible approximations to perform it. Being interested in nearest neighbour effects, we follow Niemeijer and van Leeuwen in choosing the simplest two-cell cluster approximation.

For the cluster ij in figure 1, the new two-cell interaction, $h'(\mu_i, \mu_j)$, is given by

$$\exp(h'(\mu_i, \mu_j)) = z_0(\mu_i) z_0(\mu_j) \langle \exp(h(S_i^2, S_j^1) + h(S_i^3, S_j^1)) \rangle \tag{4}$$

where

$$z_0(\mu_i) = \sum_{\{S_i^\alpha\}}{}' \exp(h(S_i^1, S_i^2) + h(S_i^2, S_i^3) + h(S_i^3, S_i^1)) \tag{5}$$

the sum being only over configurations of the three spins in cell i consistent with the given μ_i, and the average $\langle A \rangle$ is with the Boltzmann factors of equation (5) and with similarly restricted sums.

The main new feature of the transformation, beyond the $S = \frac{1}{2}$ calculation of Niemeijer and van Leeuwen, is the definition of the cell spin, equation (2). It has recently been emphasised by van Leeuwen (1975) that this transformation should preserve all the important symmetries of the problem. The same spirit was adopted by Berker and Wortis (1976), who chose to preserve the symmetries of the Ising model limit $M \to \infty$ and of the Griffiths–Ising limit $K \to 0$. We choose to emphasise the Potts model symmetry, i.e. the invariance of the Hamiltonian under permutations of the three possible values of the spin variable (± 1 or 0) when $K = \frac{1}{3}L = -\frac{1}{2}M$. To preserve this symmetry, we follow Harris *et al* (1975) in using the majority rule: $\mu_i(S_i^1, S_i^2, S_i^3)$ is equal to S_i^α if any two spins are equal to S_i^α. If all three spins are different from each other, then we have two possible procedures:

(A1) $\mu_i(S_i^1, S_i^2, S_i^3) = S_i^1$, independent of S_i^2 and S_i^3,

or

(A2) assign one third of the contributions that these configurations make to the sums in equation (3) to each possible value of μ_i.

There are many other possible transformations. For example, the simple assignment used by Niemeijer and van Leeuwen (1974), namely

$$\mu_i(S_i^1, S_i^2, S_i^3) = \operatorname{sgn}(S_i^1 + S_i^2 + S_i^3),$$

where $\operatorname{sgn}(0) = 0$, does not give equal weights to the states $\mu_i = \pm 1$ or 0. This leads to an unreasonable fixed point structure.

Using the assignments (A1) or (A2), we are now ready to calculate the new pair interaction $h'(\mu_i, \mu_j)$. The identity

$$\exp[KS_iS_j + LS_i^2S_j^2 + M(S_i^2 + S_j^2)] = 1 + C_1S_iS_j + C_2S_i^2S_j^2 + C_3(S_i^2 + S_j^2) \tag{6}$$

where

$$C_1 = \exp(2M + L)\sinh(K)$$

$$C_2 = 1 - 2\exp(M) + \exp(2M + L)\cosh(K)$$

$$C_3 = \exp(M) - 1$$

is used, to yield

$$\exp[h'(\mu_i, \mu_j)] = X_0 + X_1\mu_i\mu_j + X_2\mu_i^2\mu_j^2 + X_3\mu_i^2 + X_4\mu_j^2, \tag{7}$$

where the various coefficients are listed in table 1. The calculation involves averages like $\langle (S_i^\alpha)^k (S_i^\beta)^l \rangle$, which are listed in table 2.

One should note the asymmetry between the coefficients of μ_i^2 and μ_j^2 in equation (7). In order to eliminate this asymmetry we follow Harris *et al* (1975), and average over all possible triangle pairs.

Our new cell spin Hamiltonian is $H'(K', L', M')$ and we find the new parameters K', L', M' by transforming the right-hand side of (7) back to exponential form. The

Table 1. Coefficients for scheme A1, (for A2 replace C_1 and C_2 by C etc P_1 and P_2 by P_3.)

$$X = \exp(2M+L)\sinh(K)$$

$$V = 1-2\exp(M)+\exp(2M+L)\cosh(K)$$

$$Y = \exp(M)-1$$

$$X_0 = B^2 - X^2W_2E_1 + 2VE_1E_2 + 2VYE_1(W_2+E_2) + 2YB(E_1+E_2) + V^2W_2E_1 + Y^2[B(E_1+W_2)+2E_1E_2]$$

$$X_1 = 2XC_1C_2 + 2XVC_1F + 2XYC_1(F+C_2)$$

$$X_2 = (A-B)^2 + X^2(P_2+W_2)(D_1-E_1) + 2V(D_1-E_1)(D_2-E_2) + 2VY(H_2-W_2+D_2-E_2)(D_1-E_1) + 2Y(A-B)(D_1-E_1+D_2-E_2) + V^2(H_2-W_2)(D_1-E_1) + Y^2[(A-B)(H_2-W_2+D_1-E_1)+2(D_1-E_1)(D_2-E_2)]$$

$$X_3 = (A-B)B + X^2E_1(P_2+W_2) + 2VE_1(D_2-E_2) + 2VYE_1(H_2-W_2+D_2-E_2) + 2Y[(D_2-E_2)B+(A-B)E_1] + V^2E_1(H_2-W_2) + Y^2[(H_2-W_2)B+2(D_2-E_2)E_1+(A-B)E_1]$$

$$X_4 = (A-B)B - X^2(D_1-E_1)W_2 + 2V(D_1-E_1)E_2 + 2VY(D_2-E_2)W_2 + E_2 + 2Y[(A-B)E_2+(D_1-E_1)B] + V^2(D_1-E_1)W_2 + Y^2[(A-B)W_2+2(D_1-E_1)E_2+(D_1-E_1)B]$$

$$Y_0 = X_0$$

$$Y_1 = X_0+X_1+X_2+X_3+X_4$$

$$Y_2 = X_0-X_1+X_2+X_3+X_4$$

$$Y_3 = X_0+X_3$$

$$Y_4 = X_0+X_4$$

values are

$$e^{K'} = (Y_1/Y_2)^{1/2} \qquad e^{L'} = [Y_1Y_2Y_0^2/(Y_3)^2(Y_4)^2]^{1/2} \qquad e^{M'} = (Y_3/Y_0)^a(Y_4/Y_0)^b \tag{8}$$

where $a = \frac{1}{3}$, $b = \frac{2}{3}$. The Y coefficients are given in table 1.

For scheme (A1) we must also calculate the recursion relations for the cluster ik of figure 1, then average to obtain

$$\begin{aligned} K' &= (K'_{ij}+2K'_{ik})/3 \\ L' &= (L'_{ij}+2L'_{ik})/3 \\ M' &= (M'_{ij}+2M'_{ik})/3. \end{aligned} \tag{9}$$

The Hamiltonian of equation (1) (denoted H1) may also be written in the more usual form (H2), where the interaction S_i^2 is associated with sites rather than bonds

$$-\beta\mathcal{H} = \sum_{\langle ij\rangle}(KS_iS_j + LS_i^2S_j^2) + N\sum_i (S_i^2) \tag{10}$$

where $N = zM$, z being the coordination number.

For the two-cell cluster approximation, the effective coordination number z_{eff}, is somewhat less than $z = 6$ for the triangular lattice. This means that the values of a

Table 2. Averages.

	Scheme A1	Scheme A2
$z_0 = \langle 1 \rangle = (A-B)\mu_i^2 + B$	$A = P+3Q+3R+2U$ $B = 1+2U+6S$	
$\langle S_i^1 \rangle = C_1\mu_i$ $\langle S_i^\beta \rangle = C_2\mu_i \quad \beta = 2,3$	$C_1 = P+Q+2R+2U$ $C_2 = P+Q+2R-U$	$C = P+Q+2R$
$\langle (S_i^1)^2 \rangle = (D_1 - E_1)\mu_i^2 + E_1$	$D_1 = P+3Q+2R+2U$ $E_1 = 2S$	$D = P+3Q+2E+\frac{4}{3}U$
$\langle (S_i^\beta)^2 \rangle = (D_2 - E_2)\mu_i^2 + E_2$	$D_2 = P+3Q+2R+U$ $E_2 = 2S+2U$	$E = 2S+\frac{4}{3}U$
$\langle S_i^1 S_i^\beta \rangle = (P_1 + W_1)\mu_i^2 - W_1$	$P_1 = P-Q+R-U$ $W_1 = 0$	$P_3 = P-Q-R-\frac{2}{3}U$
$\langle S_i^2 S_i^3 \rangle = (P_2 + W_2)\mu_i^2 - W$	$P_2 = P-Q+R$ $W_2 = 2U$	$W = \frac{2}{3}U$
$\langle (S_i^1)^2 (S_i^\beta)^2 \rangle = (H_1 - W_1)\mu_i^2 + W_1$ $\langle (S_i^2)^2 (S_i^3)^2 \rangle = (H_2 - W_2)\mu_i^2 + W_2$	$H_1 = P+3Q+R+U$ $H_2 = P+3Q+R$	$H = P+3Q+R+\frac{2}{3}U$
$\langle S_i^1 (S_i^\beta)^2 + S_i^\beta (S_i^1)^2 \rangle = 2F$ $\langle S_i^2 (S_i^3)^2 + S_i^3 (S_i^2)^2 \rangle = 2F$	$F = P+Q+R$	

$P = \exp(3(K+L+2M)$ from configurations where $S_i^\beta = S_i^\gamma = S_i^\delta \neq 0$

$Q = \exp(-K+3L+6M)$ from configurations where $S_i^\beta = S_i^\gamma = -S_i^\delta \neq 0$

$R = \exp(K+L+4M)$ from configurations where $S_i^\beta = S_i^\gamma \neq S_i^\delta = 0$

$S = \exp(2M)$ from configurations where $0 = S_i^\beta = S_i^\gamma \neq S_i^\delta$

$U = \exp(-K+L+4M)$ from configurations where $S_i^\beta \neq S_i^\gamma \neq S_i^\delta$

and b given in equation (8) are no longer justified, and we have

$$a + b = z/z_{\text{eff}}.$$

We expect the Potts fixed point for this Hamiltonian to occur at

$$K = \tfrac{1}{3}L = -N/2z_{\text{eff}}.$$

For the Griffiths Ising spin-$\frac{1}{2}$ behaviour we expect $K^* = 0$,

$$L^* = 4K_{\text{I}}^* \qquad \text{and} \qquad N^* = -2z_{\text{eff}}K_{\text{I}}^* - \ln 2$$

for H2 (for H1 we expect the last to be replaced by $M^* = -2K_{\text{I}}^* - \ln 2$), where K_{I}^* denotes the fixed point value of K for the usual spin-$\frac{1}{2}$ Ising model ($L = M = 0$). We have also carried out a one-parameter calculation directly on the three-state Potts Hamiltonian (H3) which is H1 with $K = \frac{1}{3}L = -\frac{1}{2}M$.

In this case the sums in equations (4) and (5) must be taken over fixed values of both μ_i and μ_j. We use the relationship

$$e^{K\delta_{ij}} = 1 + (e^K - 1)\delta_{ij}$$

and require the averages $\langle \delta_{s_i^\alpha s_j^\beta} \rangle$, $\langle (\delta_{s_i^\beta s_j^\alpha})(\delta_{s_i^\gamma s_j^\alpha}) \rangle$ etc (see table 3).

The recursion relations are identical to those for H1 with the Potts coefficients.

Table 3. δ function averages (scheme A1).

$\langle \delta_{s_i^\beta s_j^1} \rangle = A\delta_{\mu_i\mu_j} + B(1-\delta_{\mu_i\mu_j})$	$A = e^{6\kappa} + 8e^{4\kappa} + 2e^{3\kappa} + 10e^{\kappa}$ $B = 2e^{4\kappa} + e^{3\kappa} + 9e^{2\kappa} + 7e^{\kappa} + 2$
$\langle \delta_{s_i^\beta s_j^\gamma} \rangle = C\delta_{\mu_i\mu_j} + D(1-\delta_{\mu_i\mu_j})$	$C = e^{6\kappa} + 6e^{4\kappa} + 2e^{3\kappa} + 8e^{2\kappa} + 2e^{\kappa}$ $D = e^{4\kappa} + 2e^{2\kappa}$
$\langle (\delta_{s_i^\beta s_j^1})(\delta_{s_i^\gamma s_j^1}) \rangle = F\delta_{\mu_i\mu_j} + G(1-\delta_{\mu_i\mu_j})$	$F = e^{6\kappa} + 8e^{4\kappa} + 18e^{2\kappa} + 4e^{\kappa} + 2$ $G = 2e^{4\kappa} + 2e^{3\kappa} + 9e^{2\kappa} + 10e^{\kappa} + 1$
$\langle (\delta_{s_i^\beta s_j^1})(\delta_{s_i^\beta s_j^\gamma}) \rangle = H\delta_{\mu_i\mu_j} + Z(1-\delta_{\mu_i\mu_j})$	$H = e^{6\kappa} + 6e^{4\kappa} + 8e^{2\kappa}$ $Z = e^{4\kappa} + e^{3\kappa} + 2e^{2\kappa} + 2e^{\kappa}$

where $\beta, \gamma = 2, 3$

3. Fixed points and their exponents

The global scheme for our two different Hamiltonians and two different averaging schemes is very similar. It is also similar to that obtained by Berker and Wortis (1976). It thus would appear that these overall features are independent of both the lattice structure and the exact nature of the transformation, the former to be expected from universality and the latter to be hoped for. We cannot compare locations of our fixed points with those of Berker and Wortis (1976) and Burkhardt (1976) since their work was for the square lattice but we will compare eigenvalues and the proximity of the Potts fixed point to its axis. The locations of the higher order fixed points and their exponents can be found in tables 4 and 5, and we now consider some regions that are of special interest.

Table 4. Critical points and exponents for H1 and H3.

	(K, L, M)	(y_2, y_4, y_6)
Ising	$(0{\cdot}365, -, \infty)$	$(0{\cdot}79, -, -)$
Griffiths	$(0, 1{\cdot}69, -1{\cdot}01)$	$(3{\cdot}4, 0{\cdot}9, -1{\cdot}3)$
Tricritical	$(0{\cdot}88, 2, 0{\cdot}29, -0{\cdot}519)$	$(3{\cdot}1, 0{\cdot}6, -0{\cdot}7)$
Potts	$(0{\cdot}447, 3K, -2K)$	$(3{\cdot}3, 1{\cdot}0, 0{\cdot}5)$
One-parameter Potts (H3)	$(0{\cdot}447, 3K, -2K)$	$(-, 1{\cdot}0, -)$

Table 5. Critical points and exponents for H2 ($a + b = 1{\cdot}6$).

	(K, L, N)	(y_2, y_4, y_6)
Ising	$(0{\cdot}365, -, \infty)$	$(0{\cdot}79, -, -)$
Griffiths	$(0, 2{\cdot}09, -3{\cdot}76)$	$(2{\cdot}1, 1{\cdot}0, -1{\cdot}1)$
Tricritical	$(0{\cdot}954, 0{\cdot}017, -1{\cdot}34)$	$(2{\cdot}0, 0{\cdot}4, -1{\cdot}1)$
Potts	$(0{\cdot}376, 1{\cdot}80, -3{\cdot}47)$	$(2{\cdot}1, 1{\cdot}0, 0{\cdot}6)$

3.1. The $L = 0$ plane

This plane is equivalent to the Blume–Capel model (Blume 1966, Capel 1966, 1967a,b). The critical line in this plane is shown in figure 2. The general shape of the line is similar to the mean field result and that obtained by Berker and Wortis (1976). In figure 3 we present this plane again and note that all the points (with spin-1 Hamiltonians) along the segment TS flow to the spin-$\frac{1}{2}$ fixed point $K^* = 0{\cdot}365$, $M^* = \infty$. This value of K^* is the same coupling constant as Niemeijer and van Leeuwen found for their two-cell cluster approximation and in fact our recursion relations reduce exactly to theirs in the limit $M \to \infty$. This proves, within our renormalisation group, the universal equivalence of the spin-1 and the spin-$\frac{1}{2}$ Ising models.

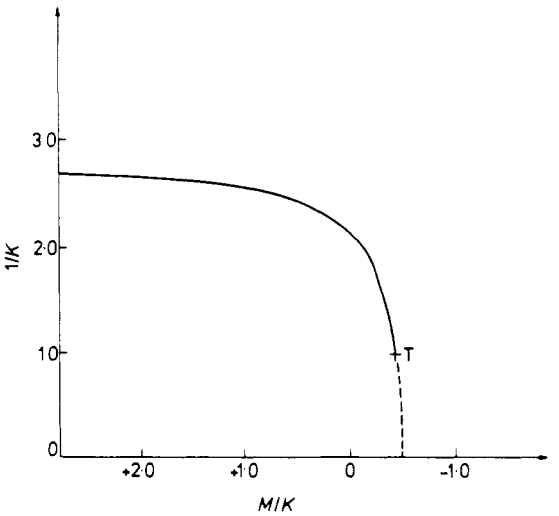

Figure 2. The Blume–Capel phase diagram ($L = 0$). The full and broken curves correspond to second and first order transitions. T is the initial point from which flows go to the tricritical fixed point.

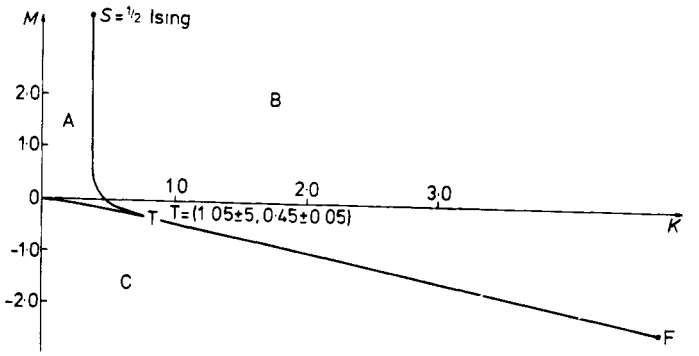

Figure 3. The Blume–Capel phase diagram. The points (with spin-1 Hamiltonians) along TS flow to the $S = \frac{1}{2}$ Ising fixed point S. From T we flow to the tricritical point. The points in region A flow to $(0, 0, \infty)$, those in C to $(0, 0, -\infty)$ and those in B to $(\infty, -\infty, \infty)$. The points along TF flow to $(\infty, \infty, -\infty)$.

The points below T on TF flow to an infinite fixed point at $(\infty, \infty, -\infty)$ and from T we flow to the tricritical point. In this plane we also find trivial fixed points at

$$(0, 0, 0), \quad (0, 0, \infty) \quad \text{and} \quad (0, 0, -\infty).$$

These correspond to the three trivial points of Berker and Wortis (1976). The points in region A (see figure 3) flow to $(0, 0, \infty)$ those in C to $(0, 0, -\infty)$ and those in region B flow to $(\infty, -\infty, \infty)$.

3.2. The $K = 0$ plane

In the $K = 0$ plane we find the Griffiths critical point $K^* = 0$ exactly but, for both H1 and H2, L^* is somewhat greater than predicted. (See tables 4 and 5.) The differences between the predicted and actual values of L^*/M^* for H1 and H2 suggest a z_{eff} of around 3·6 which, as expected, is lower than the $z = 6$ for a triangular lattice (as discussed in § 2).

We now consider the eigenvalues of this point. For y_T we obtain values of 1·0 and 0·9 for H2 and H1 respectively which can be compared with the exact Ising value of 1·000. Our spin-$\frac{1}{2}$ fixed point has $y_T = 0{\cdot}79$ and Berker and Wortis (1976) obtain 0·9419 from their transformation, which is adjusted to the Griffiths symmetry. Our spin-$\frac{1}{2}$ value is in agreement with the two-cell cluster calculation of Niemeijer and van Leeuwen (1976). They do not achieve a value near 1·0 until they consider a five-cell cluster. This suggests that one should not necessarily expect better than 20% accuracy for y_T from a two-cluster approximation, and we return to this point below. There is also a fixed point at $(0, \infty, -\infty)$.

3.3. The tricritical point

In table 6 we compare our tricritical exponents with other calculations. Considering the approximations involved, the values in the table are not inconsistent with universality of tricritical points at $d = 2$, both with respect to the lattice structure and with respect to the type of tricriticality (metamagnet or BEG).

Table 6. The tricritical exponents.

	y_2	y_4	y_6
Our H1	3·1	0·6	−0·7
Our H2	2·0	0·4	−1·1
$PSRG_0$[a]	1·92	0·7192	−0·6654
$PSRG_v$[a]	1·8373	0·9181	−0·6875
Metamagnet[b]	1·852	0·652	—
Burkhardt[c]	1·7966	0·7966	—
ϵ expansion[d]	1·968	1·2	−2

[a] Berker and Wortis (1976), [b] Nienhuis and Nauenberg (1976), [c] Burkhardt (1976), [d] Stephen and McCauley (1973), Chang *et al* (1974), Tuthill *et al* (1975), Wegner (1975).

3.4. The Potts tricritical point

As discussed in § 2 we also carried out a one-parameter renormalisation group calculation for this model which gave us a very good check on the recursion relations

for H1. We find that points on the Potts line $(K, 3K, -2K)$ flow to $(0, 0, 0)$ for $K < 0{\cdot}447$ and to $(\infty, \infty, -\infty)$ for $K > 0{\cdot}447$. We note that for H1 and H3 we obtain the Potts fixed point exactly on the Potts axis. Our value for the eigenvalue y_T agrees to two significant figures with the series value of Straley and Fisher (1975) (see table 7)†. These results indicate that we have been successful in our aim (§ 2) to preserve the Potts model symmetries. For H3 our fixed point is close to the Potts axis.

Table 7. The Potts exponents.

	y_2	y_4	y_6
Our H1	3·3	1·0	0·5
Our H2	2·1	1·0	0·6
Our H3	—	1·0	—
$PSRG_0$[a]	1·9416	0·8327	0·4645
$PSRG_v$[a]	1·8704	1·1063	0·5248
Dasgupta[b]	1·8715	1·1806	0·4570
Burkhardt[c]	1·87	1·202	0·4620
Series[d]	—	1·0	—

[a] Berker and Wortis (1976), [b] Dasgupta (1976), [c] Burkhardt *et al* (1976), [d] Straley and Fisher (1975).

3.5. The first-order fixed points

These have been listed above in their various regions of phase space. They correspond to those of Berker and Wortis (1976), except that we do not distinguish between the two different portions of their 'F_3T_0PL' plane.

4. Conclusion

Overall, our calculation can be said to have been successful. We have studied two slightly different Hamiltonians, each with two different cell assignments in the renormalisation group transformation and carried out a one-parameter transformation on the Potts model.

We find the same qualitative phase diagram for all these combinations. The main features included the Potts symmetry line and fixed point, the tricritical fixed point and the Griffiths–Ising spin-$\frac{1}{2}$ fixed point and we also found fixed points at $T = 0$ and $T = \infty$. It was also noticed that the spin-1 models flowed to the spin-$\frac{1}{2}$ fixed point thus demonstrating the universality of Ising critical behaviour. It also appears that within the limits of our calculation tricritical behaviour is universal at $d = 2$.

We note that none of our transformations contain any adjustable parameters that would enable us to artificially improve our results. The two averaging schemes give almost identical results everywhere, however the two different Hamiltonians do not. We note that Berker and Wortis (1976) only considered a Hamiltonian analogous to our H2 and could not exactly retain all the Potts symmetries. This was also the case with our H2. We were able to retain these symmetries exactly with H1 and H3 and

† Note however that other series calculations yield $y_T \simeq 1{\cdot}2$ (Kim and Joseph 1975).

obtain very good Potts eigenvalue results for all our Hamiltonians. Thus it appears that problems with the Potts fixed point are solved by using H1. However, H1 does not give very good results for the y_2 exponents of the Griffiths–Ising spin-$\frac{1}{2}$ and tricritical fixed points.

The locations of the various fixed points also are different for the two Hamiltonians but we have no independent comparison available for the triangular lattice.

We do not feel that the solution to these problems lies in adding adjustable parameters to obtain numerical values since as noted in § 3 above, we are generalising a two-cell cluster approximation, that only gives a 20% accuracy for an important eigenvalue. It would seem that the best way to improve the numerical results would be to carry out a calculation with larger clusters; thus increasing the effective coordination number.

We predict that as the effective coordination number approaches $z = 6$ we would obtain better agreement between results from H1 and H2. We would also hope that these results would be even closer to the exact values for all the fixed points.

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